Frontiers of Network Science
Fall 2023

Class 9: Scale Free Networks and Barabasi Model II
(Chapters 4-5 in Textbook)

Boleslaw Szymanski

based on slides by Albert-László Barabási & Roberta Sinatra

The meaning of scale-free

Scale-free networks: Definition

Definition:

Networks with a power law tail in their degree distribution are called ‘scale-free networks’

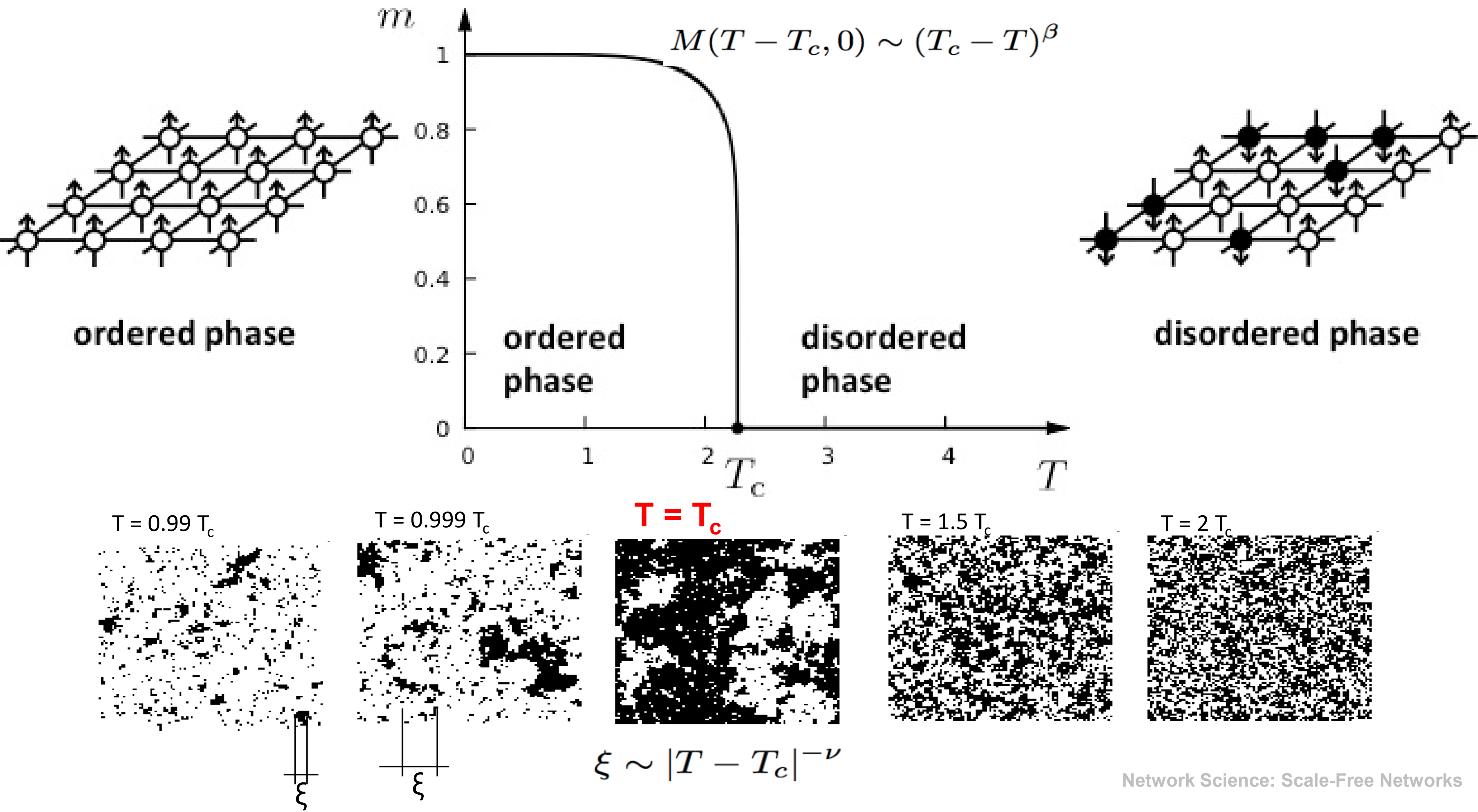
Where does the name come from?

Critical Phenomena and scale-invariance

(a detour)

Slides after **Dante R. Chialvo**

Phase transitions in complex systems I: Magnetism



CRITICAL PHENOMENA

- Correlation length diverges at the critical point: the whole system is correlated!

- **Scale invariance**: there is no characteristic scale for the fluctuation **(scale-free behavior).**

- **Universality**: exponents are independent of the system's details.

Divergences in scale-free distributions

$$P(k) = Ck^{-\gamma} \qquad k = [k_{\min}, \infty) \qquad \int_{k_{\min}}^{\infty} P(k)dk = 1 \qquad C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1)k_{\min}^{\gamma-1}$$

$$P(k) = (\gamma - 1)k_{\min}^{\gamma-1}k^{-\gamma}$$

$$< k^m >= \int_{k_{\min}}^{\infty} k^m P(k) dk \qquad < k^m >= (\gamma - 1)k_{\min}^{\gamma-1} \int_{k_{\min}}^{\infty} k^{m-\gamma}\, dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{\gamma-1} \left[k^{m-\gamma+1} \right]_{k_{\min}}^{\infty}$$

If m-γ+1<0: $$< k^m >= -\frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{m}$$

If m-γ+1>0, the integral diverges.

For a fixed γ this means that all moments with m>γ-1 diverge.

DIVERGENCE OF THE HIGHER MOMENTS

$$< k^m >= (\gamma - 1) k_{\min}^{\gamma-1} \int_{k_{\min}}^{\infty} k^{m-\lambda}\, dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{\gamma-1} \left[k^{m-\gamma+1} \right]_{k_{\min}}^{\infty}$$

For a fixed γ this means all moments m>γ-1 diverge.

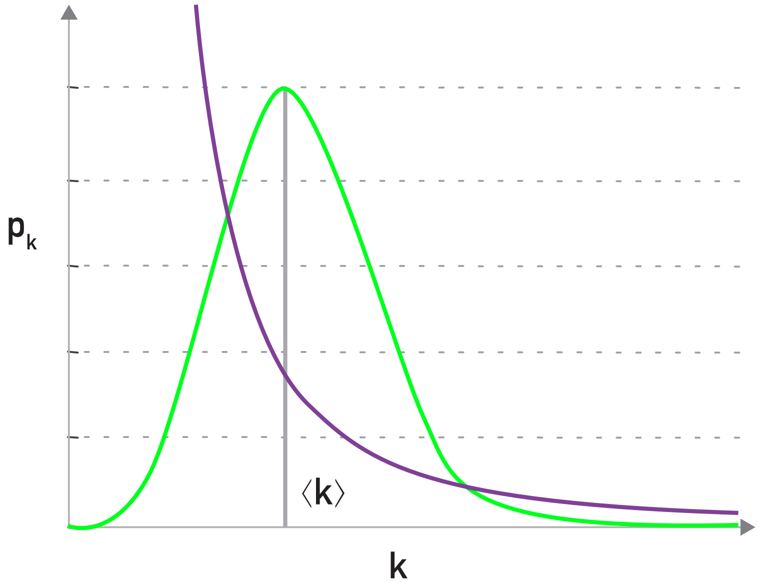
Network	N	L	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

Many degree exponents are smaller than 3

→ $<k^2>$ diverges in the N→∞ limit!!!

→ $<k>$ diverges in the N→∞ limit!!!

The meaning of scale-free



Random Network

Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$

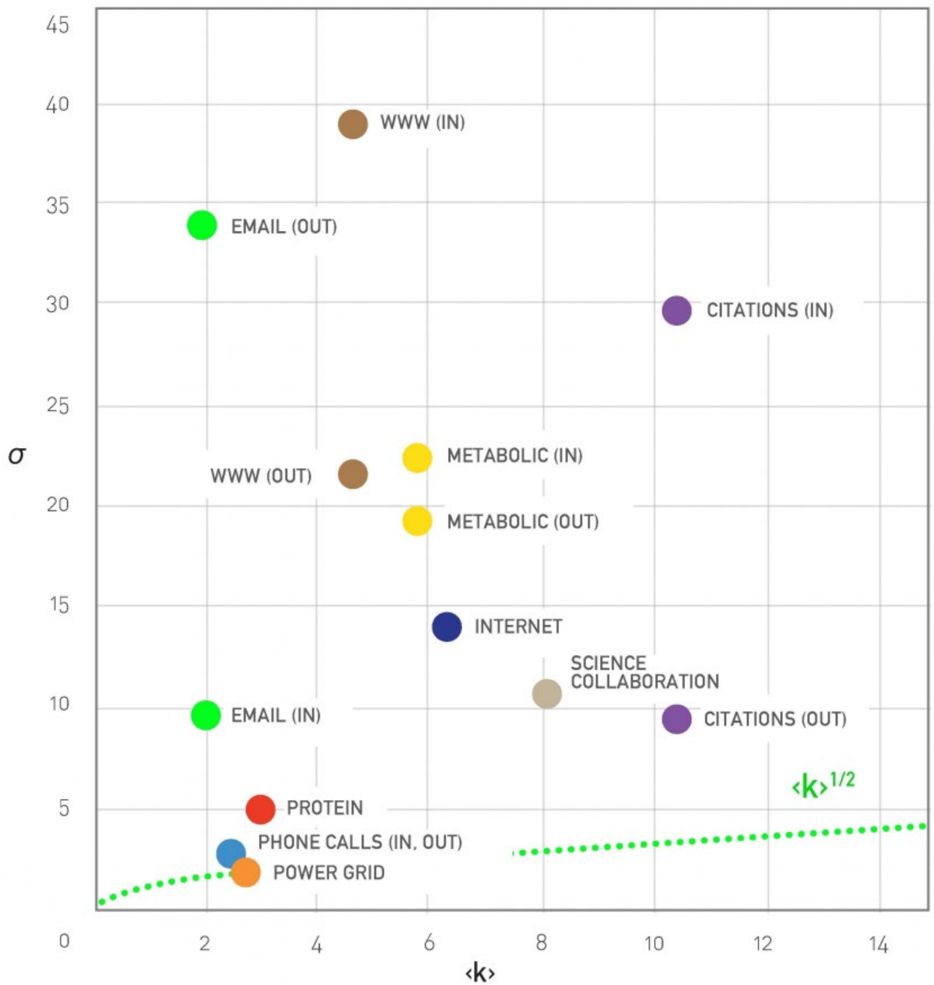
Scale: $\langle k \rangle$

Scale-Free Network

Randomly chosen node: $k = \langle k \rangle \pm \infty$

Scale: none

The meaning of scale-free

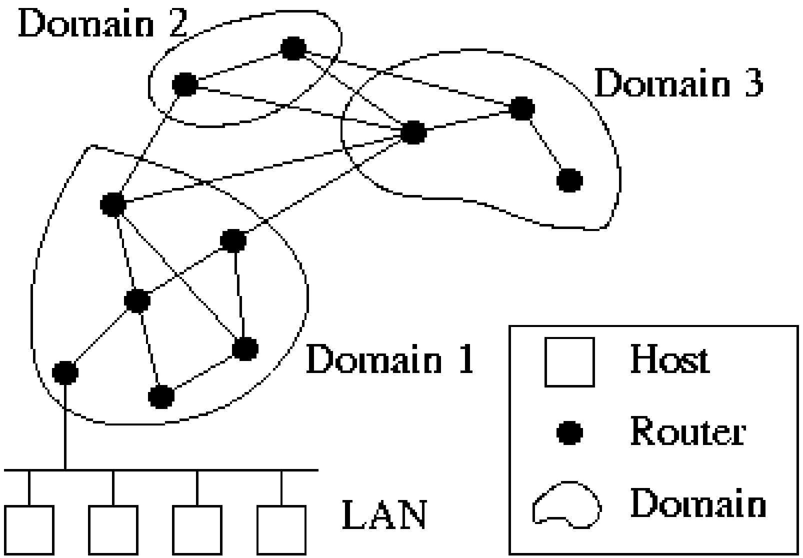


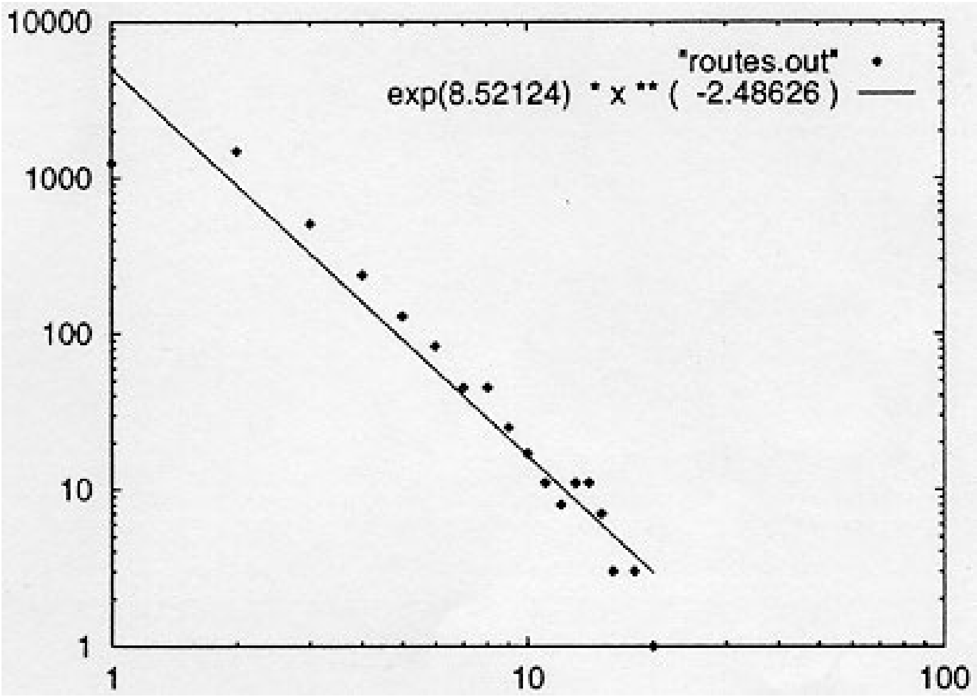
$$k = \langle k \rangle \pm \sigma_k$$

universality

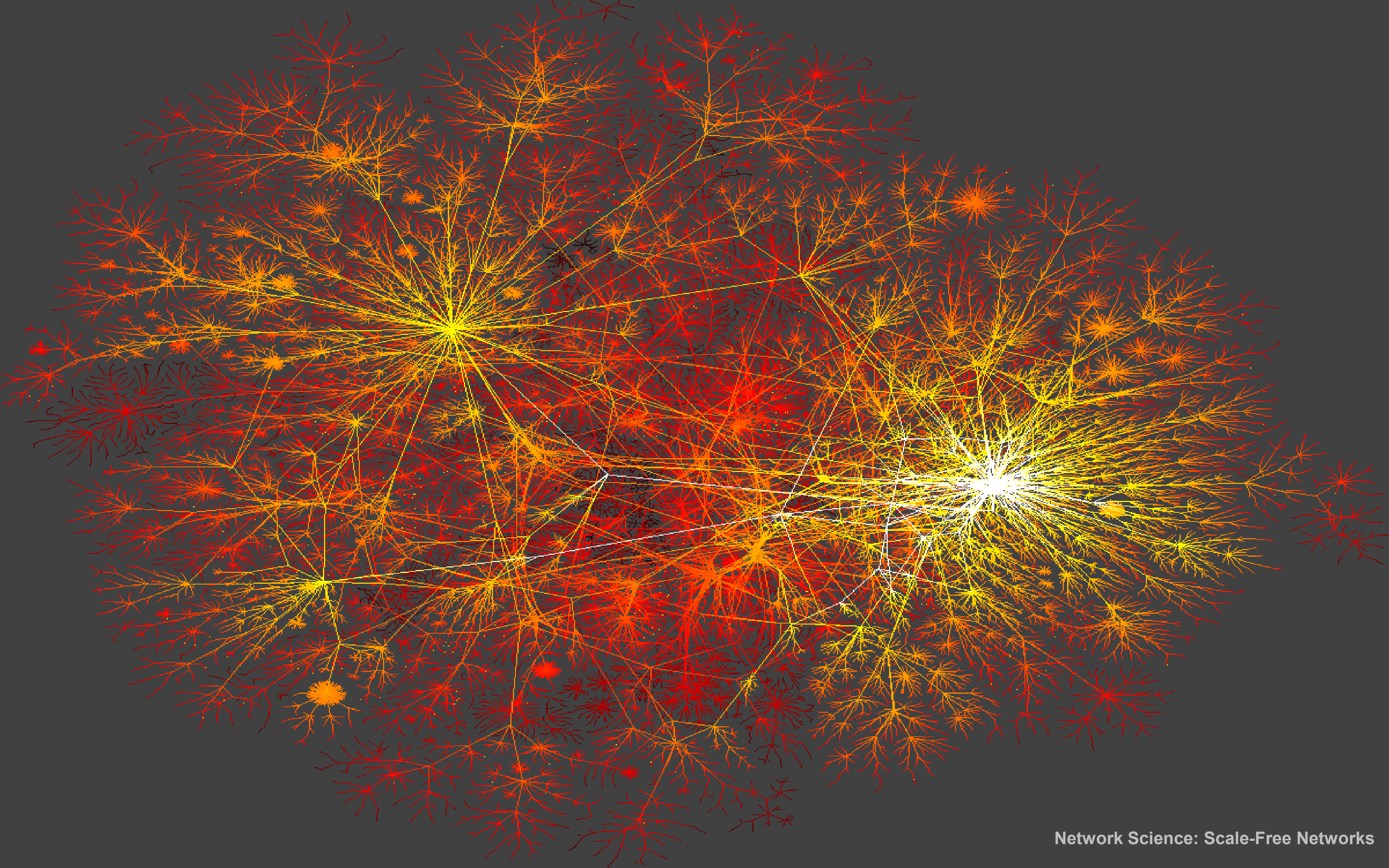
INTERNET BACKBONE

Nodes: computers, routers
Links: physical lines





(Faloutsos, Faloutsos and Faloutsos, 1999)

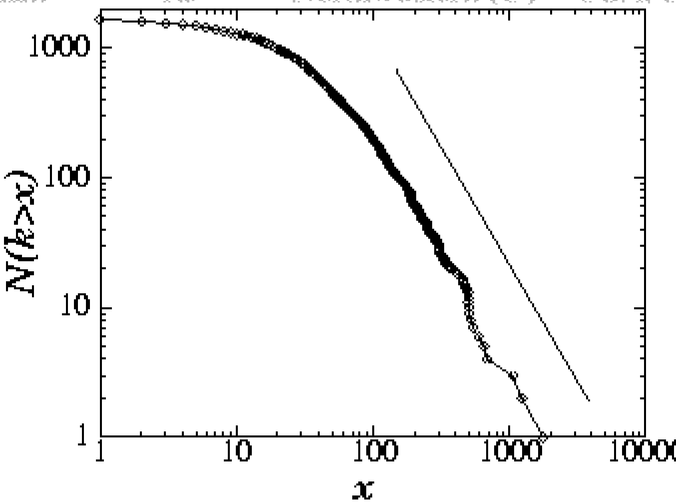


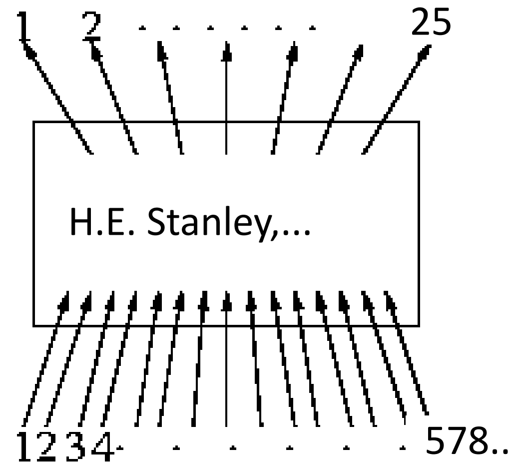
SCIENCE CITATION INDEX

Nodes: papers

Links: citations

1736 PRL papers (1988)





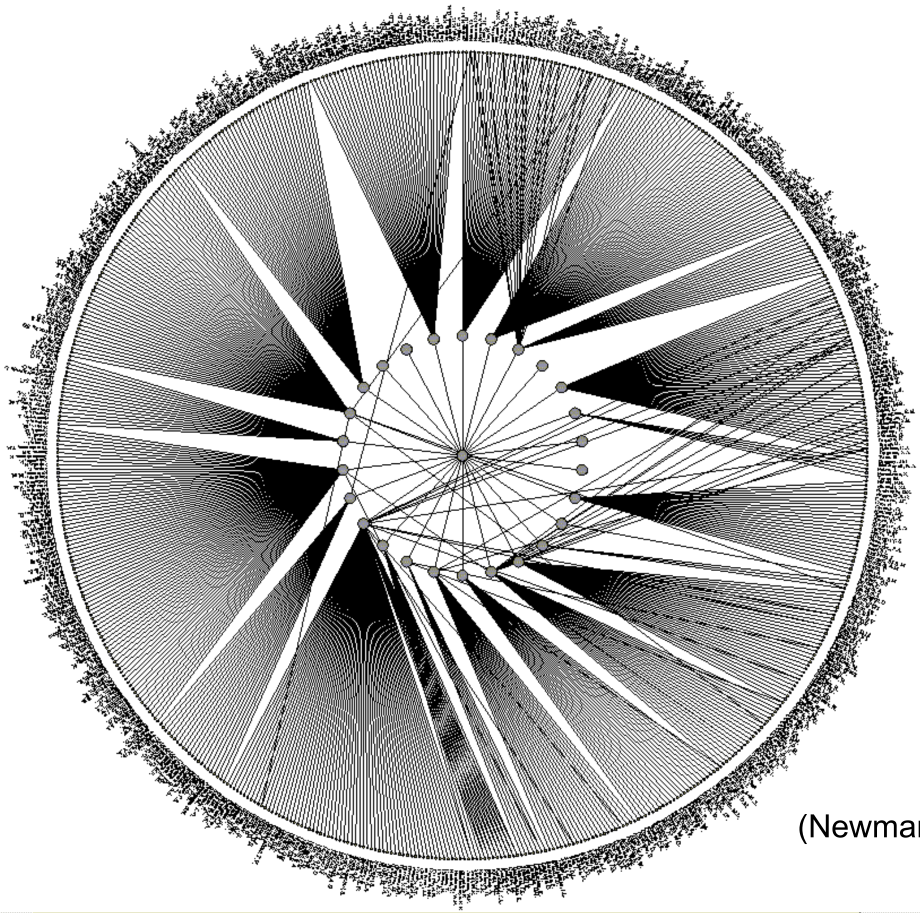
$P(k) \sim k^{-\gamma}$

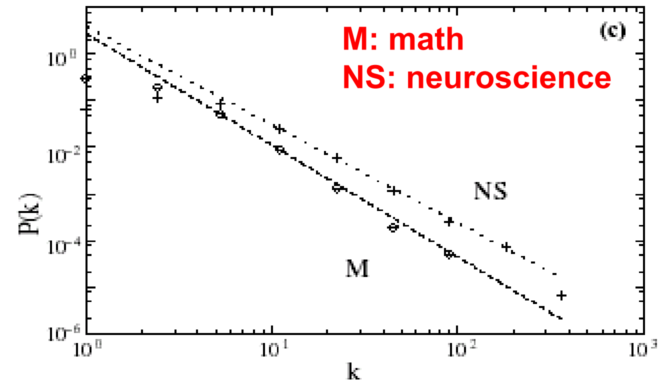
$(\gamma = 3)$

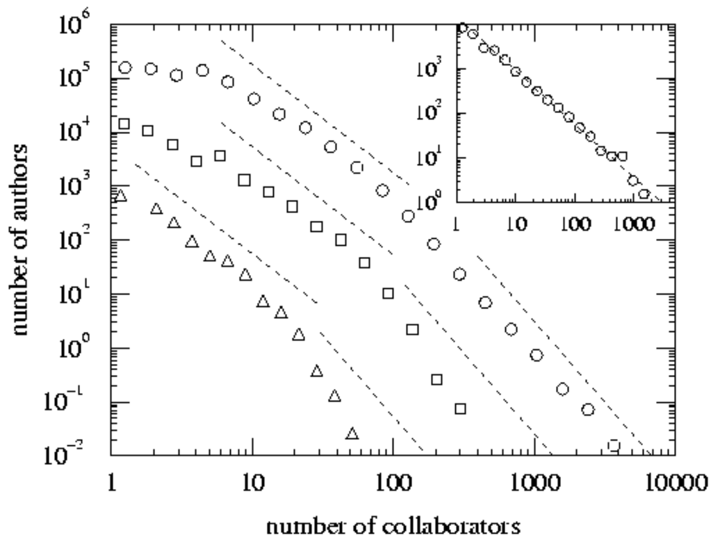
(S. Redner, 1998)

SCIENCE COAUTHORSHIP

Nodes: scientist (authors)
Links: joint publication





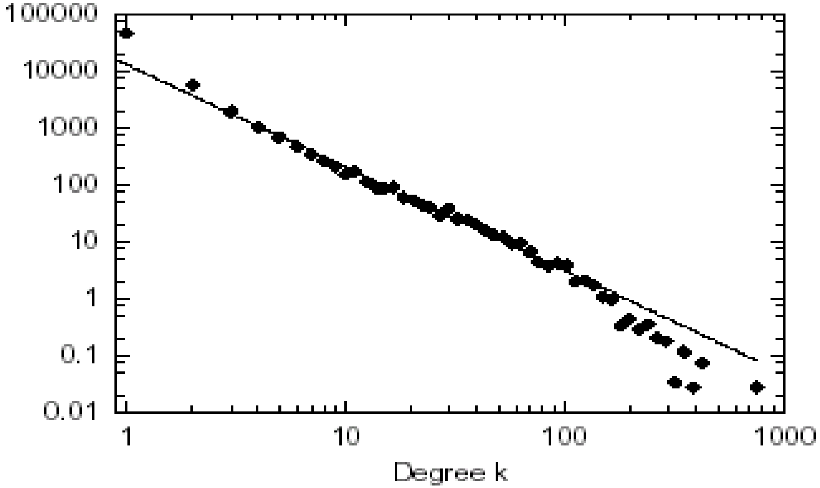


(Newman, 2000, Barabasi et al 2001)

ONLINE COMMUNITIES

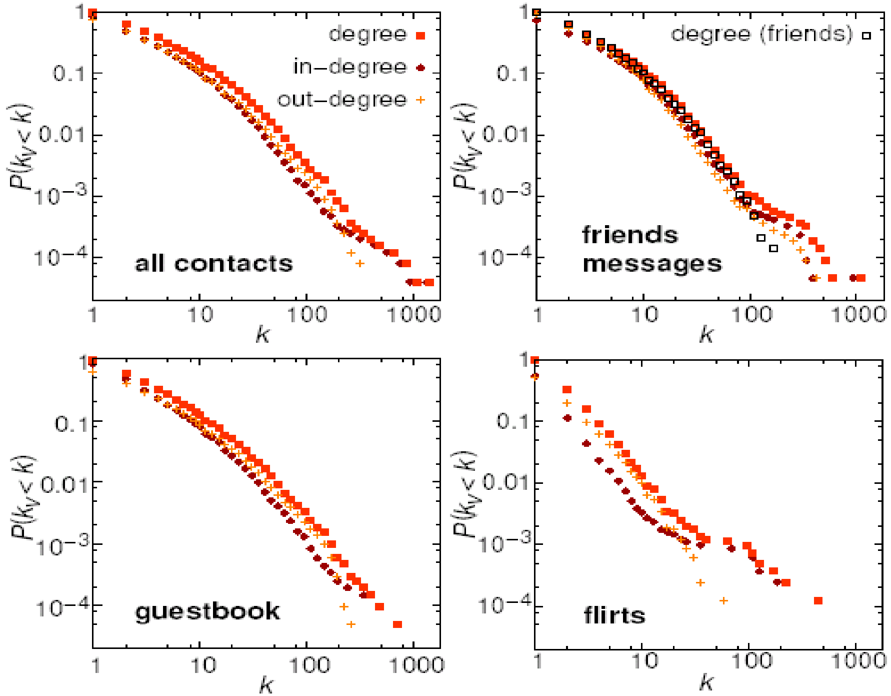
Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



Ebel, Mielsch, Bornholdtz, PRE 2002.

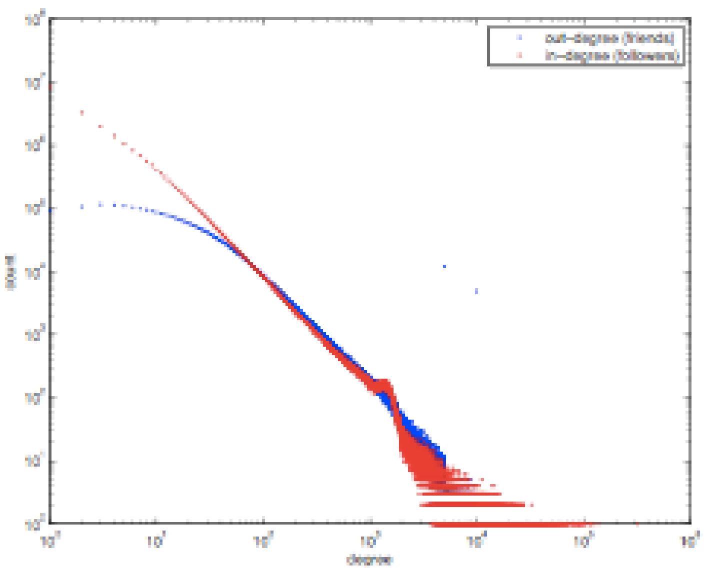
Pussokram.com online community;
512 days, 25,000 users.



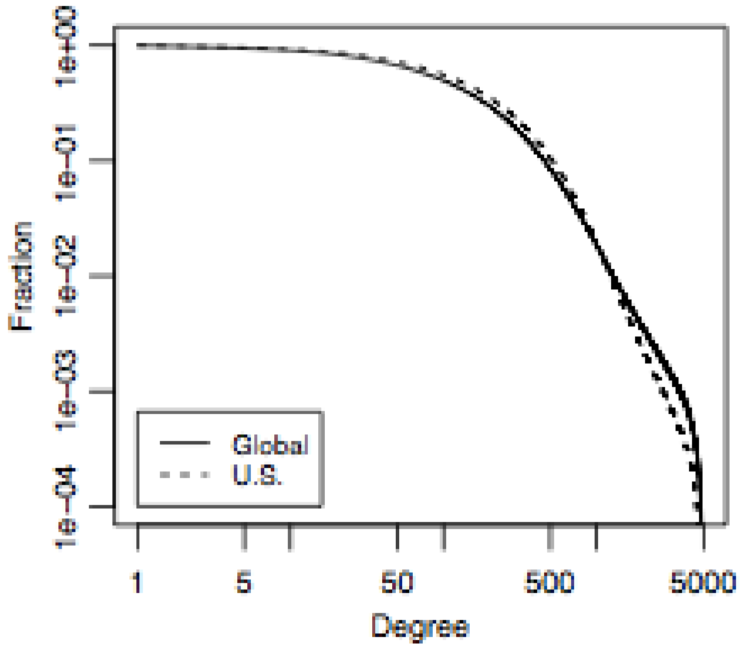
Holme, Edling, Liljeros, 2002.

ONLINE COMMUNITIES

Twitter:



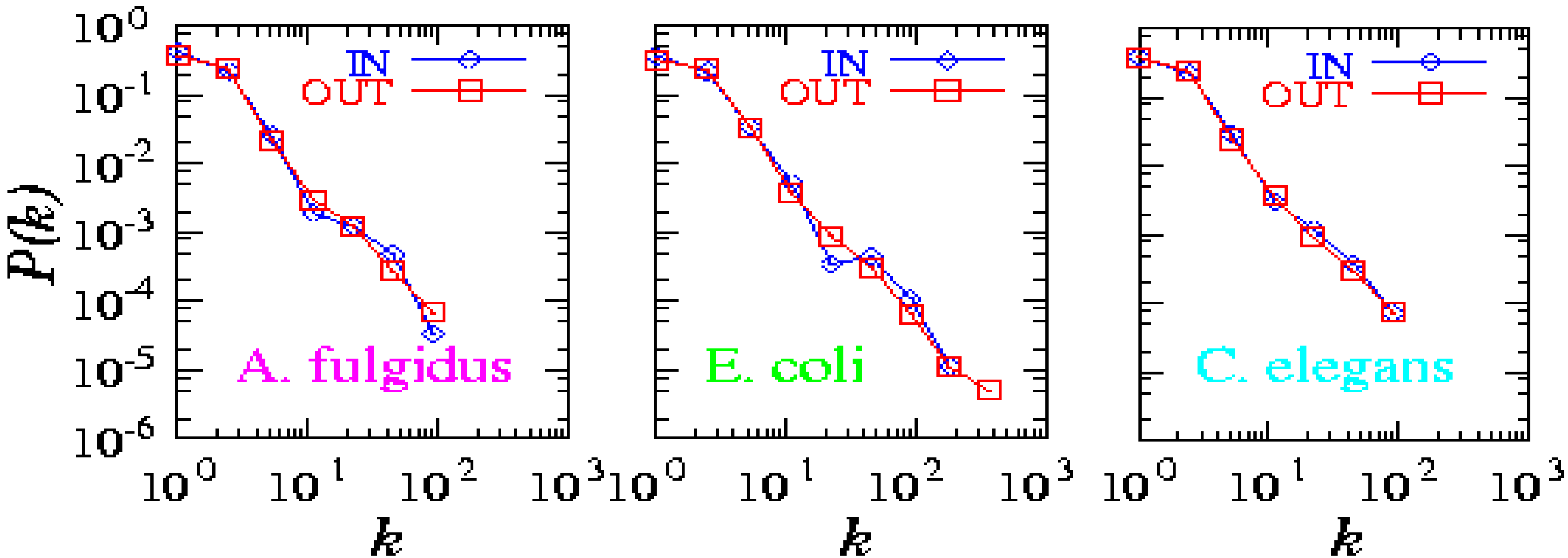
Facebook



Brian Karrer, Lars Backstrom, Cameron Marlowm 2011

Barabasi-Albert Model

METABOLIC NETWORK



Archaea

Bacteria

Eukaryotes

Organisms from all three domains of life are scale-free!

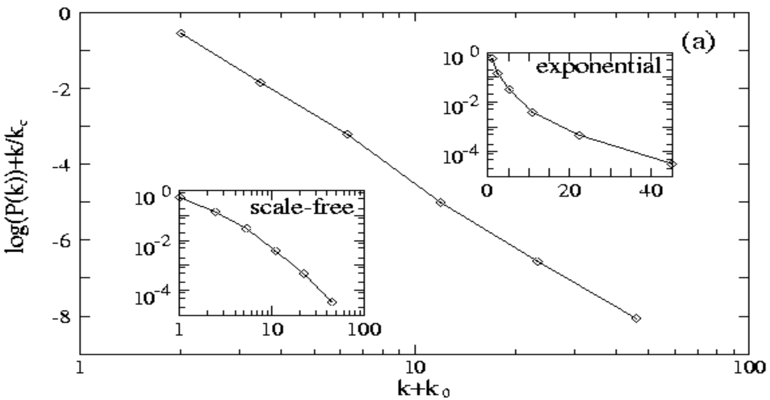
$$P_{in}(k) \approx k^{-2.2}$$

$$P_{out}(k) \approx k^{-2.2}$$

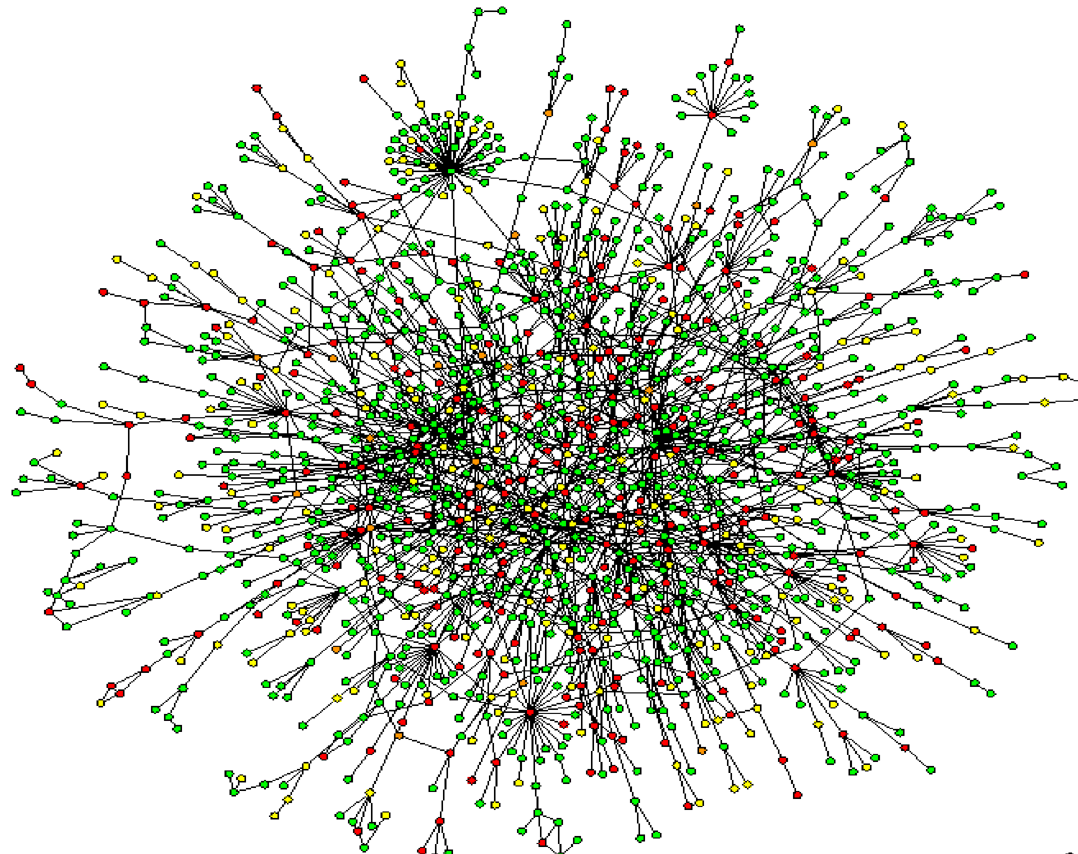
H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

TOPOLOGY OF THE PROTEIN NETWORK

Nodes: proteins
Links: physical interactions-binding

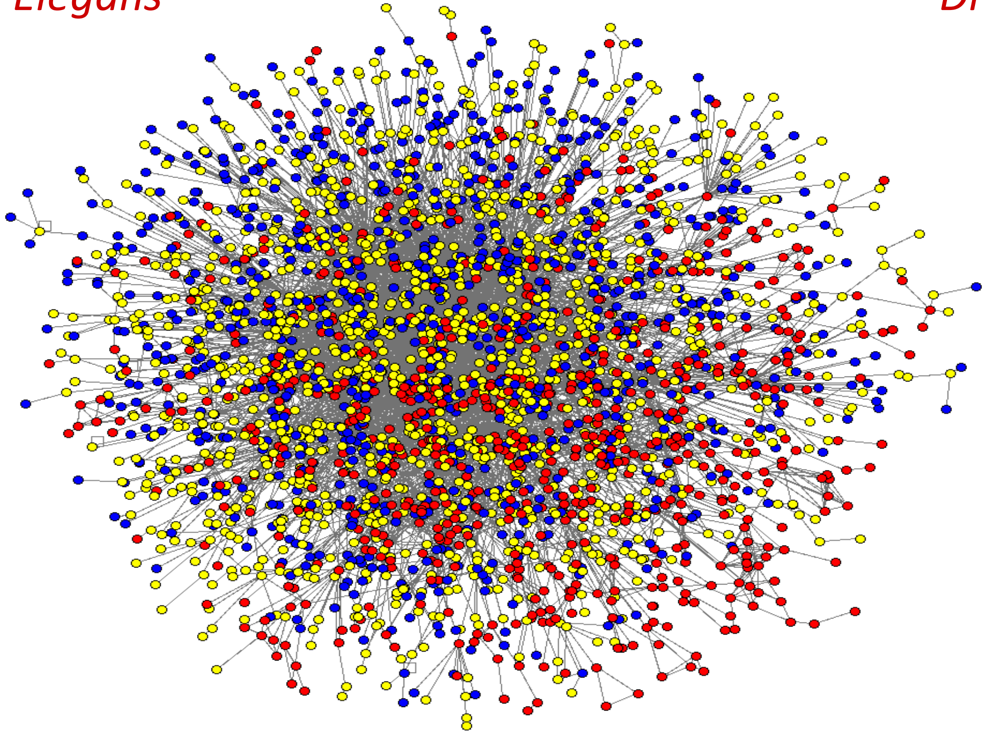


$$P(k) \sim (k+k_0)^{-\gamma} \exp\left(-\frac{k+k_0}{k_\tau}\right)$$

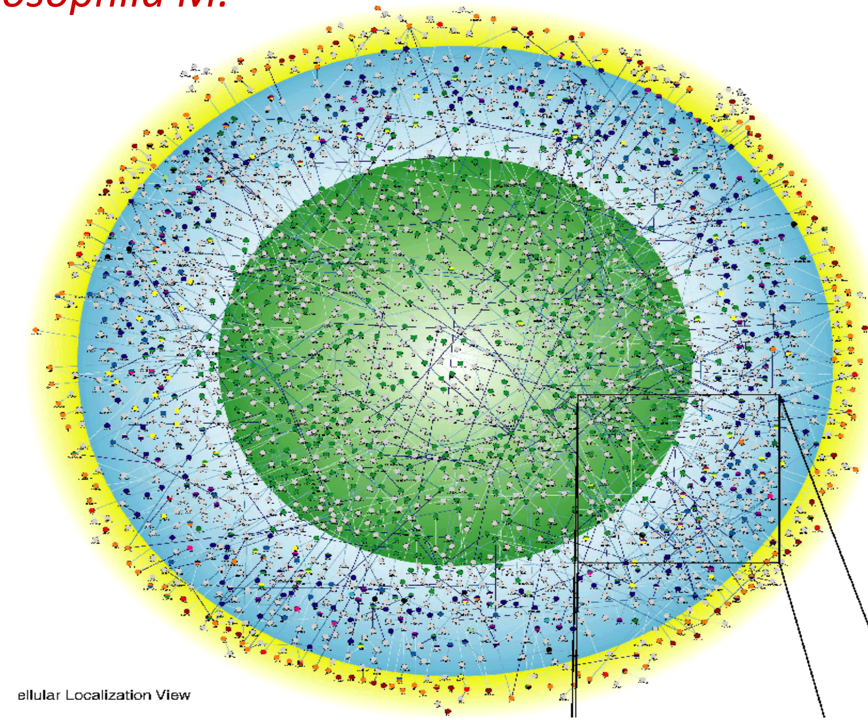


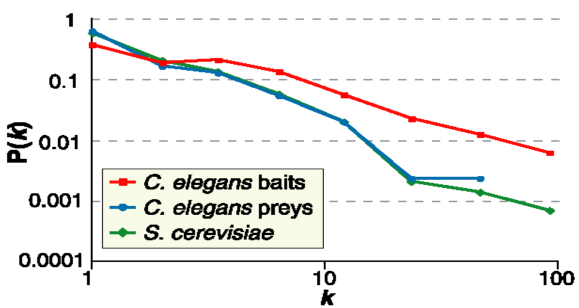
H. Jeong, S.P. Mason, A.-L. Barabasi, Z.N. Oltvai, Nature 411, 41-42 (2001)

C. Elegans

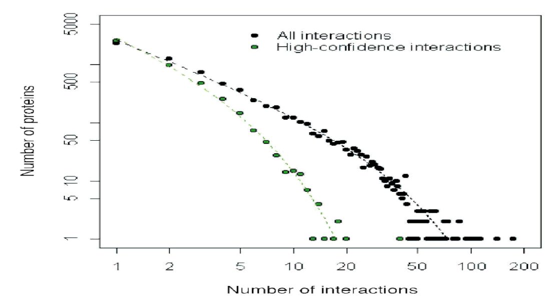


Drosophila M.





Li *et al.* Science 2004



Giot *et al.* Science 2003

Growth and preferential attachment

Barabasi-Albert model Definition

The recognition that growth and preferential attachment coexist in real networks has inspired a minimal model called the **Barabási-Albert model (BA model),** which generates scale-free networks [1], defined as follows:

We start with m_0 nodes, the links between which are chosen arbitrarily, as long as each node has at least one link. The network develops following two steps:

1. **Growth:** at each timestep we add a new node with m ($\leq m_0$) links that connect the new node to m nodes already in the network.
2. **Preferential attachment:** **t**he probability $\Pi(k)$ that a link of the new node connects to node i depends on the degree k_i as $\Pi(k_i)=k_i \sum_j k_j$

Preferential attachment is a probabilistic mechanism: a new node is free to connect to any node in the network, whether it is a hub or has a single link. However, that if a new node has a choice between a degree-two and a degree-four node, it is twice as likely that it connects to the degree-four node.

[1] A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

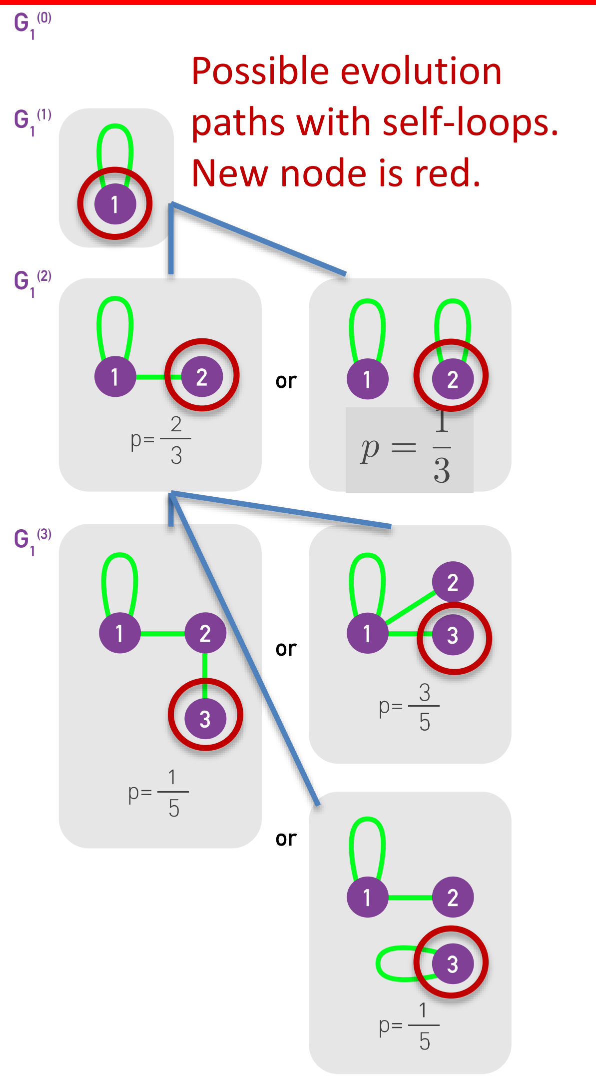
Linearized Chord Diagram

The definition of the Barabási-Albert model leaves many mathematical details open:

- It does not specify the precise initial configuration of the first m_0 nodes.
- It does not specify whether the m links assigned to a new node are added one by one, or simultaneously. This leads to potential mathematical conflicts: If the links are truly independent, they could connect to the same node i, leading to multi-links.

One possible definition with self-loops

$$p(i=s)=\begin{cases} \dfrac{k_i}{2t-1} & if\ 1 \le s \le t-1 \\ \dfrac{1}{2t-1}, & if\ s=t \end{cases}$$



Degree dynamics

Degree distribution for Barabasi-Albert model

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \qquad \beta = \frac{1}{2}$$ for $t \geq m_0+i$ and 0 otherwise as system size at t is $N=m_0+t-1$

We assume the initial m_0 nodes create a fully connected graph.

A random node j arriving at time t is with equal probability $1/N=1/(m_0+t-1)$ one of the nodes $1, 2, \ldots N$, its degree will grow with the above equation, so

$$P\left(k_j(t)\right) < k\big) = P\left(t_j > \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right) = 1 - P\left(t_j \leq \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}}\right) = 1 - \frac{m^{\frac{1}{\beta}} t}{k^{\frac{1}{\beta}}(t + m_0 - 1)}$$

For the large times t (and so large network sizes) we can replace t-1 with t above, so

$$\therefore P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^2 t}{m_o + t}\frac{1}{k^3} \sim k^{-\gamma}$$

$$\gamma = 3$$

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

Degree distribution

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \qquad \beta = \frac{1}{2}$$

$$\mathrm{P(k)} = \frac{2m^2 t}{t - t_0}\frac{1}{k^3} \sim k^{-\gamma}$$

$$\gamma = 3$$

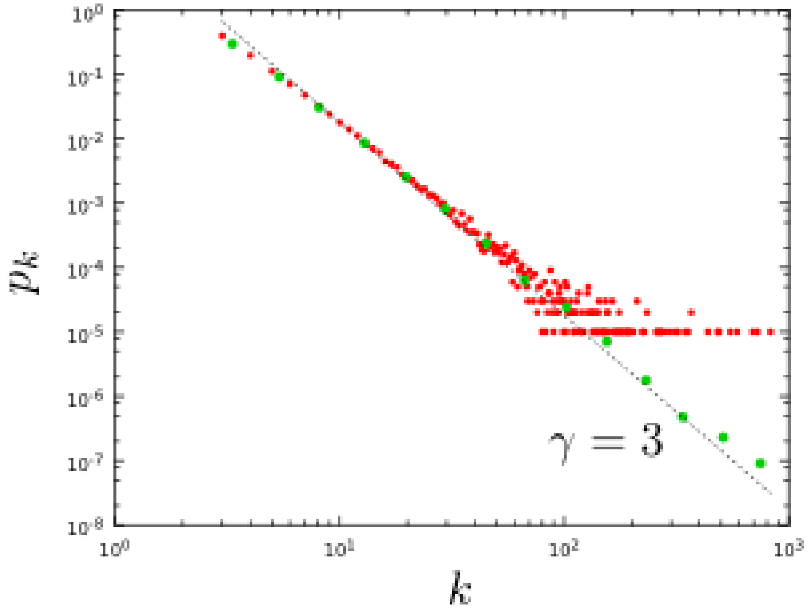
(i) The degree exponent is independent of m.

(ii) As the power-law describes systems of rather different ages and sizes, it is expected that a correct model should provide a time-independent degree distribution. Indeed, asymptotically the degree distribution of the BA model is independent of time (and of the system size N)
→ the network reaches a stationary scale-free state.

(iii) The coefficient of the power-law distribution is proportional to m^2.

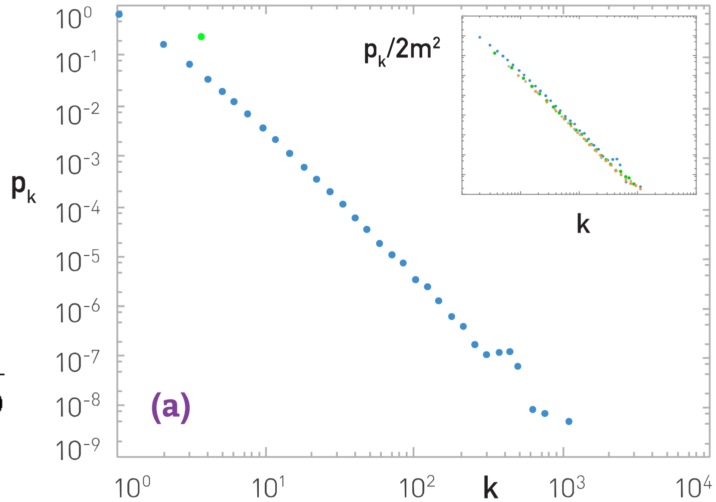
A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

Section 4

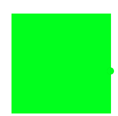


NUMERICAL SIMULATION OF THE BA MODEL

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$



(a) We generated networks with N=100,000 and m_0=m=1 (blue), 3 (green), 5 (grey), and 7 (orange). The fact that the curves are parallel to each other indicates that γ is independent of m and m_0. The slope of the purple line is -3, corresponding to the predicted degree exponent γ=3. Inset: (5.11) predicts p_k~$2m^2$, hence $p_k/2m^2$ should be independent of m. Indeed, by plotting $p_k/2m^2$ vs. k, the data points shown in the main plot collapse into a single curve.



(b)

(b) The Barabási-Albert model predicts that p_k is independent of N. To test this we plot p_k for N = 50,000 (blue), 100,000 (green), and 200,000 (grey), with m_0=m=3. The obtained p_k are practically indistinguishable, indicating that the degree distribution is stationary, i.e. independent of time and system size.

NUMERICAL SIMULATION OF THE BA MODEL

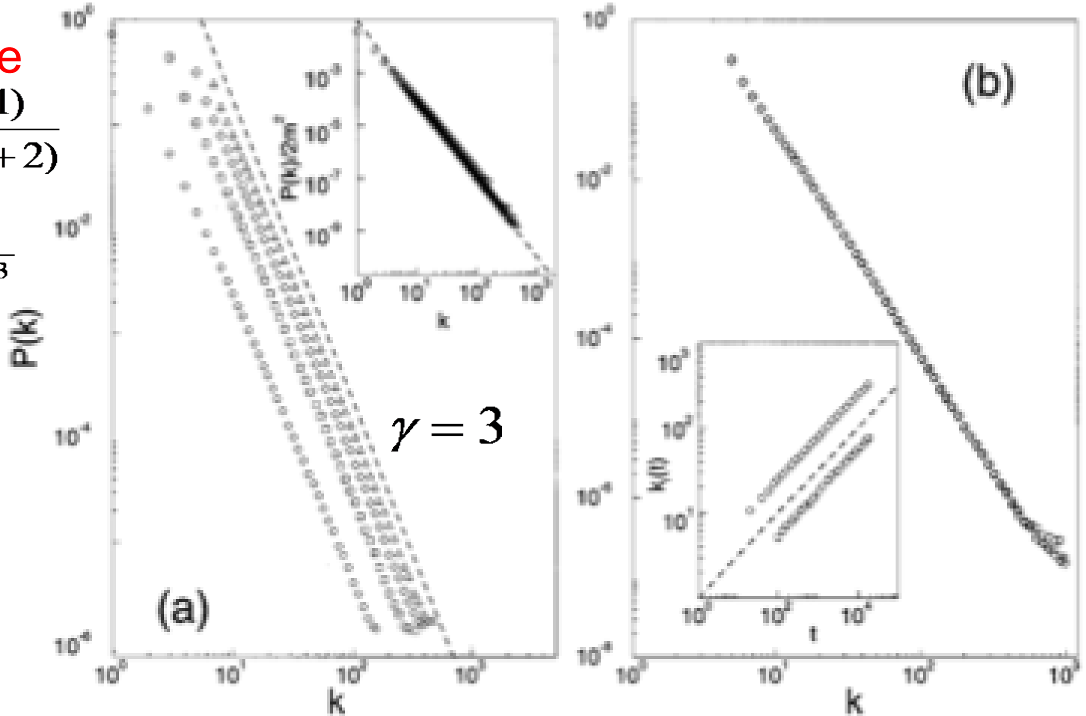
m-dependence

$$P(k)=\frac{2m(m+1)}{k(k+1)(k+2)}$$

$$P(k)=\frac{2m^2t}{m_o+t}\frac{1}{k^3}$$

m=1,3,5,7

N=100,000;150,000;200,000



Stationarity:

P(k) independent of *N*

Insert:

degree dynamics

$$k_i(t)=m\left(\frac{t}{t_i}\right)^{\beta}\qquad \beta=\frac{1}{2}$$

FIG. 21. Numerical simulations of network evolution: (a) Degree distribution of the Barabási-Albert model, with $N=m_0+t=300\,000$ and ○, $m_0=m=1$; □, $m_0=m=3$; ◇, $m_0=m=5$; and △, $m_0=m=7$. The slope of the dashed line is $\gamma=2.9$, providing the best fit to the data. The inset shows the rescaled distribution (see text) $P(k)/2m^2$ for the same values of m, the slope of the dashed line being $\gamma=3$; (b) $P(k)$ for $m_0=m=5$ and various system sizes, ○, $N=100\,000$; □, $N=150\,000$; ◇, $N=200\,000$. The inset shows the time evolution for the degree of two vertices, added to the system at $t_1=5$ and $t_2=95$. Here $m_0=m=5$, and the dashed line has slope 0.5, as predicted by Eq. (81). After Barabási, Albert, and Jeong (1999).

The mean field theory offers the correct scaling, BUT it provides the wrong coefficient of the degree distribution.

So assymptotically it is correct ($k \rightarrow \infty$), but not correct in details (particularly for small k).

To fix it, we need to calculate P(k) exactly, which we will do next using a rate equation based approach.

MFT - Degree Distribution: Rate Equation

$$< N(k,t) >= tP(K,t)$$ Number of nodes with degree *k* at time *t*.

Since at each timestep we add one node, we have N=t (total number of nodes = number of timesteps)

$$\Pi(k) = \frac{k}{\sum_j k_j} = \frac{k}{2mt}$$

2m: each node adds *m* links, but each link contributed to the degree of 2 nodes

Number of links added to degree *k* nodes after the arrival of a new node:

$$\underbrace{\frac{k}{2mt}}_{\text{Preferential attachment}} \times \overbrace{NP(k,t)}^{\text{Total number of k-nodes}} \times \underbrace{m}_{\text{New node adds m new links to other nodes}} = \frac{k}{2}P(k,t)$$

Nr. of degree *k-1* nodes that acquire a new link, becoming degree *k*

$$\frac{k-1}{2}P(k-1,t)$$

Nr. of degree *k* nodes that acquire a new link, becoming degree *k+1*

$$\frac{k}{2}P(k,t)$$

$$\underbrace{(N+1)P(k,t+1)}_{\text{\# k-nodes at time t+1}} = \underbrace{NP(k,t)}_{\text{\# k-nodes at time t}} + \underbrace{\frac{k-1}{2}P(k-1,t)}_{\text{Gain of k-nodes via k-1} \rightarrow \text{k}} - \underbrace{\frac{k}{2}P(k,t)}_{\text{Loss of k-nodes via k} \rightarrow \text{k+1}}$$

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272**, 173 (1999)

MFT - Degree Distribution: Rate Equation

$$(N+1)P(k,t+1) = NP(k,t) + \frac{k-1}{2}P(k-1,t) - \frac{k}{2}P(k,t)$$

- $(N+1)P(k,t+1)$: # k-nodes at time t+1
- $NP(k,t)$: # k-nodes at time t
- $\frac{k-1}{2}P(k-1,t)$: Gain of k-nodes via k-1→ k
- $\frac{k}{2}P(k,t)$: Loss of k-nodes via k→ k+1

We do not have *k=0,1,...,m-1* nodes in the network (each node arrives with degree *m*)
→ We need a separate equation for degree *m* modes

$$(N+1)P(m,t+1) = NP(m,t) + 1 - \frac{m}{2}P(m,t)$$

- $(N+1)P(m,t+1)$: # m-nodes at time t+1
- $NP(m,t)$: # m-nodes at time t
- 1: Add one m-degeree node
- $\frac{m}{2}P(m,t)$: Loss of an m-node via m→ m+1

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)

MFT - Degree Distribution: Rate Equation

$$(N+1)P(k,t+1) = NP(k,t) + \frac{k-1}{2}P(k-1,t) - \frac{k}{2}P(k,t)$$ k>m

$$(N+1)P(m,t+1) = NP(m,t) + \quad 1 \quad - \frac{m}{2}P(m,t)$$

We assume that there is a stationary state in the N=t→∞ limit, when P(k,∞)=P(k)

$$(N+1)P(k,t+1) - NP(k,t) \rightarrow NP(k,\infty) + P(k,\infty) - NP(k,\infty) = P(k,\infty) = P(k)$$

$$(N+1)P(m,t+1) - NP(m,t) \rightarrow P(m)$$

$$P(k) = \frac{k-1}{2}P(k-1) - \frac{k}{2}P(k)$$

$$P(m) = 1 - \frac{m}{2}P(m)$$

$$P(k) = \frac{k-1}{k+2}P(k-1)$$ k>m

$$P(m) = \frac{2}{2+m}$$

MFT - Degree Distribution: Rate Equation

$$P(k) = \frac{k-1}{k+2} P(k-1) \quad \rightarrow \quad P(k+1) = \frac{k}{k+2} P(k)$$

$$P(m) = \frac{2}{m+2}$$

$$P(m+1) = \frac{m}{m+3} P(m) = \frac{2m}{(m+2)(m+3)}$$

$$P(m+2) = \frac{m+1}{m+4} P(m+1) = \frac{2m(m+1)}{(m+2)(m+3)(m+4)}$$

$$P(m+3) = \frac{m+2}{m+5} P(m+2) = \frac{2m(m+1)}{(m+3)(m+4)(m+5)}$$

m+3 → *k*

...

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

$$P(k) \sim k^{-3}$$

for large *k*

Krapivsky, Redner, Leyvraz, PRL 2000
Dorogovtsev, Mendes, Samukhin, PRL 2000
Bollobas et al, Random Struc. Alg. 2001

MFT - Degree Distribution: A Pretty Caveat

Start from eq. $$P(k) = \frac{k-1}{2} P(k-1) - \frac{k}{2} P(k)$$

$$2P(k) = (k-1)P(k-1) - kP(k) = -P(k-1) - k[P(k) - P(k-1)]$$

$$2P(k) = -P(k-1) - k\frac{P(k) - P(k-1)}{k-(k-1)} = -P(k-1) - k\frac{\partial P(k)}{\partial k}$$

$$P(k) = -\frac{1}{2}\frac{\partial [kP(k)]}{\partial k}$$

Its solution is: $P(k) \sim k^{-3}$

Dorogovtsev and Mendes, 2003

All nodes follow the same growth law

$$\frac{\partial k_i}{\partial t} \propto \Pi(k_i) = A\frac{k_i}{\sum_j k_j}$$

In limit: $A\frac{k_i}{\sum_j k_j} = A\frac{k_i}{2mt}$ So:

$$m = \sum_i \frac{\Delta k_i}{dt} = \sum_i A\frac{k_i}{2mt} = A$$

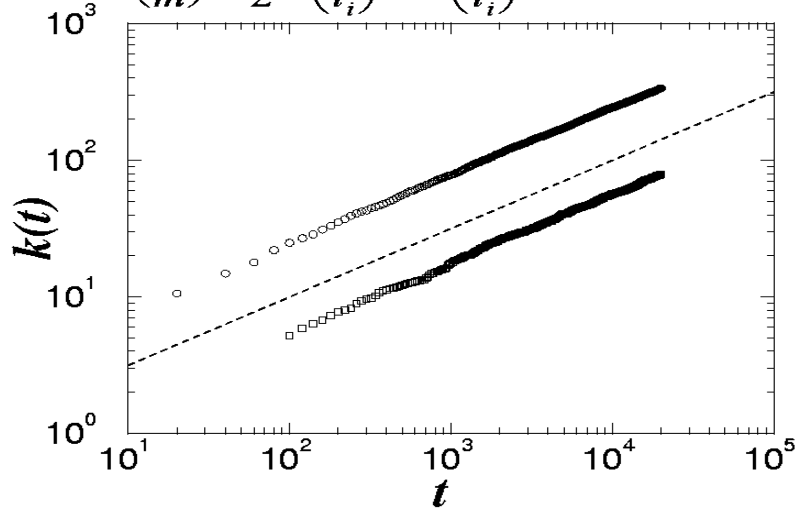
Use: $\sum_j k_j = 2m(t-1) + \frac{m_0(m_0-1)}{2}$

During a unit time (time step): $\Delta k = m$ → $A = m$

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{2t} \qquad \frac{\partial k_i}{k_i} = \frac{\partial t}{2t} \qquad \int_m^k \frac{\partial k_i}{k_i} = \int_{t_i}^t \frac{\partial t}{2t}$$

$$\ln\left(\frac{k}{m}\right) = \frac{1}{2}\ln\left(\frac{t}{t_i}\right) = \ln\left(\frac{t}{t_i}\right)^{1/2}$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \qquad \beta = \frac{1}{2}$$



β: dynamical exponent

A.-L.Barabási, R. Albert and H. Jeong, *Physica* A **272,** 173 (1999)